Dilatation symmetry of the Fokker-Planck equation and anomalous diffusion

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Based on the canonical formalism, the dilatation symmetry is implemented to the Fokker-Planck equation for the Wigner distribution that describes atomic motion in an optical lattice. This reveals the symmetry principle underlying the recent result obtained by Lutz [Phys. Rev. A **67**, 051402(R) (2003)] on the connection between anomalous transport in the optical lattice and Tsallis statistics in the high-energy regime. A condition is derived for the general linear Fokker-Planck equation to admit a nonstationary solution distribution that decays as a power law.

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Anomalous diffusion [1,2] has continuously attracted attention over the years. It can be observed, for example, in turbulent flows [3], charge transport in anomalous solids [4], dissolved micelles [5], chaotic dynamics [6], porous glasses [7], and subrecoil laser cooling [8].

Anomalous diffusion, or superdiffusion more specifically, may be a signal of scale-free non-Gaussian statistics. The process described by the Lévy distribution [9] is one such example. In the one-dimensional symmetric case, the Lévy distribution indexed by γ may be defined through the characteristic function of the stretched-exponential form $L_\gamma(x) = (1/2\pi)\int dk \exp(ikx - a|k|^\gamma)$, where $\gamma \in (0,2)$ and a is a positive constant. Excepting the Gaussian case ($\gamma \to 2-0$), $L_\gamma(x)$ decays as a power law: $L_\gamma(x) \sim |x|^{-1-\gamma}$. An important point is the following. There exists a mathematical result termed the Lévy-Gnedenko generalized central limit theorem [10], which states that, by N-fold convolution, a distribution with divergent lower moments tends to one of the Lévy stable class $\{L_\gamma(x)\}_\gamma$ in the limit $N\to\infty$ if such a limit is convergent. This is in contrast to the ordinary central limit theorem for distributions with the finite second moments in normal Gaussian statistics. Now, assume $L_\gamma(x)$ to describe a single jump in the probabilistic process in the unit time interval. After N independent jumps, one has $L_\gamma^{(N)}(x) = N^{-1/\gamma} L_\gamma(x/N^{1/\gamma})$. Identifying N with the total duration of time t, one obtains the scaling of the distribution $P(x,t) \equiv L_\gamma^{(N)}(x) = t^{-1/\gamma} P(x/t^{1/\gamma})$, implying that spreading of the distribution follows the law of superdiffusion $\sim t^{1/\gamma}$, which should be compared with the case of normal diffusion $\sim t^{1/2}$ [11]. [However, the width of $P(x,t)$ cannot be defined in terms of the variance, since the second moment of the Lévy distribution is divergent. This difficulty can be overcome by introducing the "q-expectation value." See the discussion after Eq. (6) below.]

In a recent paper [12], Lutz has reexamined the problem of anomalous transport of atoms in an optical lattice. He has considered the following generalized Fokker-Planck equation for the marginal Wigner distribution of the momentum, p, of an atom in the optical potential

$$\frac{\partial W(p,t)}{\partial t} = -\frac{\partial}{\partial p}[K(p)W(p,t)] + \frac{\partial}{\partial p}\left[D(p)\frac{\partial W(p,t)}{\partial p}\right]. \quad (1)$$

Both the drift K and the diffusion coefficient D explicitly depend on the momentum of the atom as follows:

$$K(p) = -\frac{\alpha p}{1+(p/p_c)^2}, \quad D(p) = D_0 + \frac{D_1}{1+(p/p_c)^2}. \quad (2)$$

Here, α and p_c are the damping coefficient and the capture momentum, respectively. D_0 and D_1 are constants related to fluctuations caused by the photon processes. Equation (1) can be derived, after spatial average, from the master equation for the full Wigner distribution constructed by quantum dynamics of the atom-laser interaction (see Ref. [12], and references therein). It has been noticed [12] that K and D satisfy the relation

$$\frac{K(p)}{D(p)} = \frac{d}{dp}\ln e_q[-\beta\varepsilon(p)], \quad (3)$$

where $e_q(s)$ is the "q-exponential function" defined by $e_q(s) = (1+(1-q)s)_+^{1/(1-q)}$ with the notation $(a)_+ = \max\{0,a\}$, and

$$q = 1 + \frac{2D_0}{\alpha p_c^2}, \quad \beta = \frac{\alpha}{2(D_0+D_1)}, \quad \varepsilon(p) = p^2. \quad (4)$$

In the limit $q\to 1$, $e_q(s)$ converges to the ordinary exponential function e^s. From Eq. (3), it follows that the exact stationary solution of Eq. (1) is given by [12]

$$W_q(p) = \frac{1}{Z_q(\beta)} e_q[-\beta\varepsilon(p)] \quad (5)$$

with

$$Z_q(\beta) = \int dp\, e_q[-\beta\varepsilon(p)], \quad (6)$$

which optimizes the Tsallis entropy [13,14], $S_q[W] = (1-q)^{-1}[\int dp\, W^q(p) - 1]$, under the constraints on normalization of W and the "q-expectation value" of the "energy" $\varepsilon(p)$ [15,16]: $\langle\varepsilon\rangle_q \equiv \int dp\, W^q(p)\varepsilon(p) / \int dp'\, W^q(p')$. β is related to the Lagrange multiplier associated with the q-expectation value of $\varepsilon(p)$ (see Refs. [15,16] for more details). From these discussions, we see that the entropic index, q, is determined by dynamics as in Eq. (4).

It should be mentioned that since q is larger than unity, the distribution in Eq. (5) decays as a power law. In fact, it has explicitly been shown [17] that if $q \in (5/3,3)$, many-time convolution of the distribution of the form in Eq. (5) converges to the Lévy distribution with the index $\gamma=(3-q)/(q-1)$, in accordance with the Lévy-Gnedenko generalized central limit theorem.

Before proceeding, it may be worth mentioning that the description of the Lévy-type power-law distributions has been discussed in the literature based on at least two kinds of generalizations of the ordinary Fokker-Planck equation. One is the fractional generalization [18], in which the equation becomes nonlocal in contrast to Eq. (1). The other is the nonlinear generalization [19]. There is also the combination of these two generalizations [20].

Now, emergence of an asymptotically power-law distribution from the linear Fokker-Planck equation is rather peculiar [21–23]. In the present case, the origin of the power law is in the behavior of the ratio in Eq. (3), that is,

$$\frac{K(p)}{D(p)} \sim \frac{\alpha p_c^2/D_0}{p} \tag{7}$$

in the high-energy regime. One may wonder if there is an underlying principle for the emergence of this scale-free nature. In what follows, we reveal such a principle by taking advantage of the dilatation symmetry implemented to the linear Fokker-Planck equation. As a result, without assuming stationarity, we shall obtain the condition for the solution W to be asymptotically scale-free, which turns out to be more general than Eq. (7).

Our starting point is the variational principle for the kinetic equations [24–27]. The action and the Lagrangian density, respectively, read

$$I[W,\Lambda] = \int\int dt\, dp\, \pounds(W,\Lambda,\partial W/\partial t,\partial\Lambda/\partial t,\partial W/\partial p,\partial\Lambda/\partial p), \tag{8}$$

$$\pounds = \frac{1}{2}\left(\Lambda\frac{\partial W}{\partial t} - \frac{\partial\Lambda}{\partial t}W\right) - K\frac{\partial\Lambda}{\partial t}W + D\frac{\partial\Lambda}{\partial p}\frac{\partial W}{\partial p}, \tag{9}$$

where $\Lambda(p,t)$ is an auxiliary field. Performing integration by parts and dropping the boundary terms, we see that Eq. (8) can also be expressed in the following form: $I = -\int dt\langle \partial\Lambda/\partial t + K\partial\Lambda/\partial p + \partial(D\partial\Lambda/\partial p)/\partial p\rangle$, where $\langle A\rangle$ stands for the ordinary expectation value of A: $\langle A\rangle \equiv \int dp\, A(p,t)W(p,t)$. Taking the variations of the action with respect to Λ and W, we obtain Eq. (1) and $\partial\Lambda/\partial t + K\partial\Lambda/\partial p + \partial(D\partial\Lambda/\partial p)/\partial p = 0$, respectively.

Let us proceed to developing the canonical formalism. The canonical momenta conjugate to W and Λ are given by

$$\Pi_W = \frac{\partial\pounds}{\partial(\partial W/\partial t)} = \frac{1}{2}\Lambda, \tag{10}$$

$$\Pi_\Lambda = \frac{\partial\pounds}{\partial(\partial\Lambda/\partial t)} = -\frac{1}{2}W, \tag{11}$$

respectively, leading to a pair of the constraints

$$\chi_1 = \Pi_W - \frac{1}{2}\Lambda \approx 0, \tag{12}$$

$$\chi_2 = \Pi_\Lambda + \frac{1}{2}W \approx 0, \tag{13}$$

where "$\approx$" stands for weak equality [28]. Presence of these constraints is simply due to the fact that the equations for W and Λ are of the first order in time. The basic equal-time Poisson bracket relations are

$$\{W(p,t),\Pi_W(p',t)\} = \delta(p-p'), \tag{14}$$

$$\{\Lambda(p,t),\Pi_\Lambda(p',t)\} = \delta(p-p'). \tag{15}$$

The constraints in Eqs. (12) and (13) are of the second class in Dirac's terminology [28], since $\{\chi_1(p,t),\chi_2(p',t)\} = -\delta(p-p')$ which does not vanish weakly. Then, to eliminate these second-class constraints, it is conventional to introduce the Dirac bracket [28] defined by

$$\{A(t),B(t)\}^* = \{A(t),B(t)\} - \sum_{i,j=1}^{2}\int\int dp\, dp'\{A(t),\chi_i(p,t)\} \times C_{ij}(p,p')\{\chi_j(p',t),B(t)\}, \tag{16}$$

where A and B are functionals of W and Λ. In this equation, $C_{ij}(p,p')$ are quantities satisfying the equations $\Sigma_{k=1}^{2}\int dp''\{\chi_i(p,t),\chi_k(p'',t)\}C_{kj}(p'',p') = \delta_{ij}\delta(p-p')$. In the present case, $C_{ij}(p,p')$ are given as follows: $C_{11}(p,p') = C_{22}(p,p') = 0$, $C_{12}(p,p') = -C_{21}(p,p') = \delta(p-p')$. Therefore, we obtain the basic relation

$$\{W(p,t),\Lambda(p',t)\}^* = \delta(p-p'), \tag{17}$$

which implies that W and Λ are canonically conjugate to each other with respect to the Dirac bracket.

The Hamiltonian is given by

$$H = \int dp\left(\Pi_W\frac{\partial W}{\partial t} + \Pi_\Lambda\frac{\partial\Lambda}{\partial t} - \pounds\right) = \int dp\left(K\frac{\partial\Lambda}{\partial p}W - D\frac{\partial\Lambda}{\partial p}\frac{\partial W}{\partial p}\right), \tag{18}$$

and is clearly a constant of motion. Using Eqs. (17) and (18), we can ascertain that time evolution of W, that is,

$$\frac{\partial W(p,t)}{\partial t} = \{W(p,t),H\}^*, \tag{19}$$

precisely reproduces the Fokker-Planck equation in Eq. (1). The equation for Λ is also described in a similar form.

Next, we consider the generator of the dilatation transformation

$$G = \int dp\, p\frac{\partial\Lambda}{\partial p}W. \tag{20}$$

This quantity gives rise to the following relations:

$$\{G(t),W(p,t)\}^* = \frac{\partial}{\partial p}[pW(p,t)], \tag{21}$$

$$\{G(t),\Lambda(p,t)\}^* = p\frac{\partial \Lambda(p,t)}{\partial p}. \tag{22}$$

Therefore, the finite transformations are expressed as

$$\exp\{(\ln\lambda)G(t)\}^* W(p,t)\exp\{-(\ln\lambda)G(t)\}^* = e^{(\ln\lambda)(\partial/\partial p)p}W(p,t) = \lambda W(\lambda p,t), \tag{23}$$

$$\exp\{(\ln\lambda)G(t)\}^* \Lambda(p,t)\exp\{-(\ln\lambda)G(t)\}^* = e^{(\ln\lambda)p(\partial/\partial p)}\Lambda(p,t) = \Lambda(\lambda p,t), \tag{24}$$

where λ is a positive constant and $e^{\{A\}^*}Be^{-\{A\}^*} \equiv B + \{A,B\}^* + (1/2!)\{A,\{A,B\}^*\}^* + \cdots$. Normalization of W is kept unchanged, whereas the auxiliary field Λ need not be normalized. The Dirac bracket relation in Eq. (17) is preserved, as it should be, since the transformations are canonical.

Now, we are at the position to discuss the dilatation symmetry of the system. Such a symmetry is characterized by the equation

$$\{G,H\}^* = 0. \tag{25}$$

This invariance principle may tell us under what conditions the Fokker-Planck equation in Eq. (1) admits a scale-free solution. After some calculations using Eqs. (18), (21), and (22), we obtain

$$\{G,H\}^* = \int dp\,\frac{\partial \Lambda}{\partial p}\left[KW - p\frac{\partial K}{\partial p}W - D\frac{\partial W}{\partial p} + p\frac{\partial}{\partial p}\left(D\frac{\partial W}{\partial p}\right) - D\frac{\partial}{\partial p}\left(p\frac{\partial W}{\partial p}\right)\right]. \tag{26}$$

Therefore, invariance under the dilatation transformation gives rise to the condition

$$KW - p\frac{\partial K}{\partial p}W - D\frac{\partial W}{\partial p} + p\frac{\partial}{\partial p}\left(D\frac{\partial W}{\partial p}\right) - D\frac{\partial}{\partial p}\left(p\frac{\partial W}{\partial p}\right) = 0. \tag{27}$$

We note that here W need not be stationary. We rewrite Eq. (27) as

$$\frac{\partial}{\partial p}\ln W = \frac{1}{p}\,\frac{\dfrac{\partial}{\partial p}\left(\dfrac{K}{p}\right)}{\dfrac{\partial}{\partial p}\left(\dfrac{D}{p^2}\right)}. \tag{28}$$

In what follows, we impose the condition in Eq. (27) only for large values of the momentum, since the dilatation symmetry is expected to be exact only in such a regime. The asymptotic power-law behavior of W means that

$$W(p,t) \sim \frac{a(t)}{p^{\sigma}} \tag{29}$$

holds for large values of p, where $a(t)$ is a positive function of time and the exponent σ is assumed to be a positive constant independent of time and larger than unity for the sake of normalizability. Then, $\partial \ln W/\partial p \sim -\sigma/p$, and Eq. (28) gives the condition

$$\frac{K(p)}{p} + \sigma\frac{D(p)}{p^2} \sim c, \tag{30}$$

where c is a constant. Equation (30) is our main result. Clearly, Eq. (7) can satisfy Eq. (30) in some circumstances including Eq. (2), but the latter is more general than the former. We again emphasize that Eq. (30) is free from the assumption of W to be stationary. To see the meaning of c in Eq. (30), it is necessary to consider the law of time evolution, i.e., the Fokker-Planck equation. Let us take Eq. (1) without assuming Eq. (2). Substituting Eq. (29) into Eq. (1) and using Eq. (30), we find

$$\frac{da(t)}{dt} \sim c(\sigma-1)a(t) \tag{31}$$

or its solution

$$a(t) \sim a(0)e^{c(\sigma-1)t}. \tag{32}$$

Thus, we see that c in Eq. (30) is related to the asymptotic factor in Eq. (29) as follows:

$$c \sim \frac{1}{\sigma-1}\ln\frac{a(t)}{a(0)}. \tag{33}$$

Equation (30) is an asymptotic equation that should hold for large values of p. In the specific case when K and D are given in Eq. (2), it yields the relation $-\alpha p_c^2/p^2 + \sigma D_0/p^2 \sim c$, which means that $c=0$ and accordingly the right-hand side contains only the terms that decay faster than $1/p^2$. From these, we conclude that in this specific case the asymptotic factor a in Eq. (29) is actually a constant and the exponent is given by $\sigma = \alpha p_c^2/D_0 [\equiv 2/(q-1)]$, which coincides with that in Eq. (4).

To summarize, we have developed a general method to assess the asymptotic scale-free nature of the solution of the Fokker-Planck equation. We have seen, in the special case when the distribution is stationary, how this method can reveal the symmetry principle underlying Lutz's result on the connection between anomalous transport in the optical lattice and Tsallis statistics in the high-energy regime. We have also derived the condition in the general nonstationary case, under which the solution distribution becomes scale-free.

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